Math 2A – Multivariate Calculus – Chapter 12 Test – Fall '09 Name________________________________

Show work for credit. Write all responses on separate paper. No calculators.

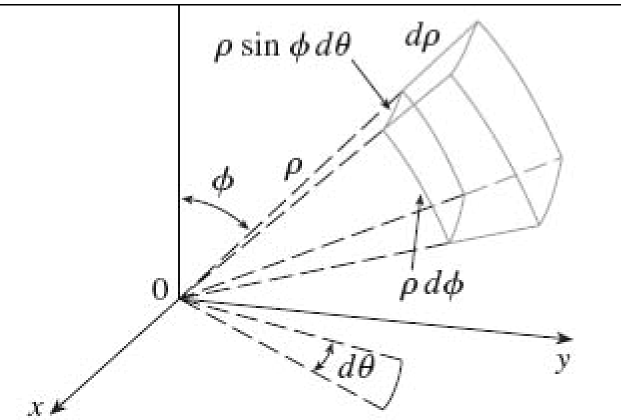
1. Approximate the double integral $\int_2^6 \int_{-4}^4 y^2 + x\,dydx$ using a Riemann sum with 2 subintervals of equal width Δx and four intervals of equal length Δy and using midpoints as sample points. Draw a diagram of the partition in the *xy* Cartesian plane and indicate the coordinates of the sample points as part of your answer. Use symmetry, as appropriate.

2. Consider the integral $\int_1^2 \int_{-x}^{x} y^2 + x\,dydx$
   a. Sketch a graph of the domain of integration in the *xy*-plane.
   b. Set up an integral(s) for to reverse the order of integration. Use symmetry, as appropriate. You don't need to evaluate the integral, unless you want to check your result.

3. Consider the diagram at right showing infinitesimals in spherical coordinates.

   a. What is an approximation the volume of the infinitesimal spherical wedge, in terms of the infinitesimals $d\rho$, $d\theta$ and $d\phi$?

   b. Use spherical coordinates to set up integrals for the (i) mass and (ii) center of mass of a solid hemisphere of radius 3 if the density at any point is proportional to its distance from the bottom.



4. This problem is about change of variables.

   Consider $\iint_D (2x+y)^2 e^{x-y}\,dA$ where *D* is region bounded by the lines $2x + y = 1$, $2x + y = 4$, $x - y = -1$, and $x - y = 1$.
   a. Determine a substitution (*u*, *v*) in terms of (*x*, *y*) that will that will simplify the double integral.
   b. Evaluate the Jacobian for this substitution.
   c. Convert the integral to one in terms of the variables *u* and *v*. Your answer should be a double integral in terms of *u* and *v* with specific limits. You don't need to evaluate the integral.

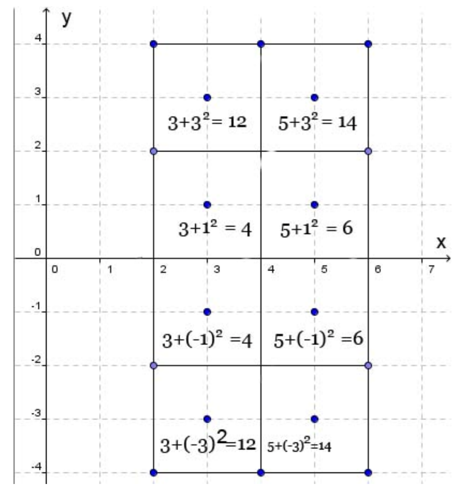
5. Set up a triple integral find the mass of tetrahedron in the first octant bounded by the plane $x + 2y + 3z = 6$ with mass density distribution is $\delta(x,y,z) = \sqrt{x^2+y^2+z^2}$.

6. Consider the ellipsoid $4z^2 + x^2 + y^2 = 4$.
   a. Set up an integral for the surface area of the ellipse. Use symmetry, as appropriate.
   b. Use the midpoint rule for double integrals with $m = n = 2$ to approximate this integral.

Math 2A – Multivariate Calculus – Chapter 12 Test Solutions – Fall '09

1. Approximate the double integral $\int_2^6 \int_{-4}^4 y^2 + x\,dydx$ using a Riemann sum with 2 subintervals of equal width Δx and four intervals of equal length Δy and using midpoints as sample points. Draw a diagram of the partition in the *xy* Cartesian plane and indicate the coordinates of the sample points as part of your answer. Use symmetry, as appropriate.

SOLN: Since there are so few sample points in the partition, and they are all integer valued, this is simple. Using symmetry across the *x*-axis we get the approximation 2(4+6+12+14)*2*2 = 288 cubic units. With sigma notation, you'd use sample points $\left(x_i^*, y_j^*\right) = \left(1+2i, -5+2j\right)$ as i = 1,2 and j = 1,2,3,4. Then you'd have



$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(y_j^*\right)^2 + x_i^*\right)\Delta y \Delta x = 4\sum_{i=1}^{2}\sum_{j=1}^{4}\left(\left(-5+2j\right)^2 + 1 + 2i\right)$$

$$= 4\sum_{i=1}^{2}\left(2\left(-3\right)^2 + 2\left(-1\right)^2 + 4\left(1+2i\right)\right)$$

$$= 4*2*20 + 16\sum_{i=1}^{2}\left(1+2i\right)$$

$$= 160 + 16\left(3+5\right) = 288$$

Once you've built the summation apparatus, it's easy to improve the computation on a computer.

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(y_j^*\right)^2 + x_i^*\right)\Delta y \Delta x = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(-4 - \frac{\Delta y}{2} + j\Delta y\right)^2 + 2 - \frac{\Delta x}{2} + i\Delta x\right)\frac{8}{n}\frac{4}{m}$$

$$= \frac{32}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\left(-4 + \frac{8\left(2j-1\right)}{2n}\right)^2 + 2 + \frac{4\left(2i-1\right)}{2m}\right)$$

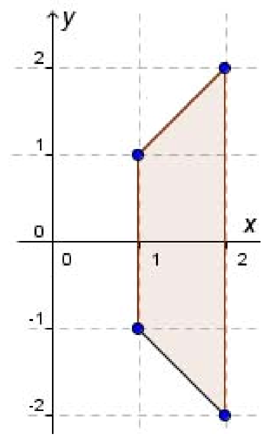
Interestingly, Mathematica simplifies

32/(m*n)*Sum[Sum[(-4+4*(2*j-1)/n)^2+2+2*(2*i-1)/m,{j,1,n}],{i,1,m}] = (128 (-4+7 n^2))/(3 n^2)

and

Integrate[Integrate[y^2+x,{x,2,6}],{y,-4,4}] = 896/3

2. Consider the integral $\int_1^2 \int_{-x}^{x} y^2 + x\,dydx$



   a. Sketch a graph of the domain of integration in the *xy*-plane.
   SOLN: The region is shown as an isosceles trapezoid in the figure at right:

   b. Set up an integral(s) for to reverse the order of integration. Use symmetry, as appropriate. You don't need to evaluate the integral, unless you want to check your result.

   SOLN: $\int_1^2 \int_{-x}^{x} y^2 + x\,dydx$

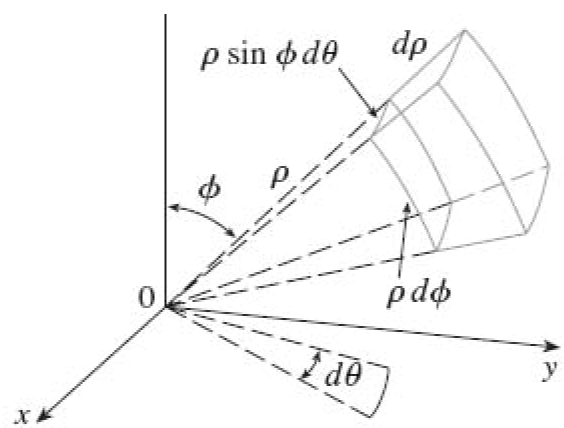
   $= 2\left(\int_0^1 \int_1^2 y^2 + x\,dxdy + \int_1^2 \int_y^2 y^2 + x\,dxdy\right)$

Here's what Mathematic gets:

$$\int_1^2 \left(\int_{-x}^{x} (x + y^2)\, d\,y\right) d\,x = 43/6$$

$$\int_0^1 \left(\int_1^2 (x + y^2)\, d\,x\right) d\,y + \int_1^2 \left(\int_y^2 (x + y^2)\, d\,x\right) d\,y = 43/12$$

3. Consider the diagram at right showing infinitesimals in spherical coordinates.



   a. What is an approximation the volume of the infinitesimal spherical wedge, in terms of the infinitesimals *dρ*, *dθ* and *dø*? SOLN:
   $dV = \rho \sin\phi d\theta \rho d\phi d\rho = \rho^2 \sin\phi d\theta d\phi d\rho$

   b. Use spherical coordinates to set up integrals for the (i) mass and (ii) center of mass of a solid hemisphere of radius 3 if the density at any point is proportional to its distance from the bottom.
   SOLN: The mass is

$$m = \iiint_{\Re} dV = k\int_0^3 \int_0^{2\theta} \int_0^{\pi/2} (\rho\cos\phi)\rho^2 \sin\phi d\theta d\phi d\rho = k\int_0^3 \int_0^{2\theta} \int_0^{\pi/2} \rho^3 \cos\phi \sin\phi d\theta d\phi d\rho$$

   Since the mass distribution is symmetric about the *z*-axis, the *x* and *y* coordinates of the center of mass are both 0. The *z*-coordinate of the center of mass is $\overline{z} = \frac{M_{xy}}{m} = \frac{k}{m}\int_0^3 \int_0^{2\theta} \int_0^{\pi/2} \rho^4 \cos^2\phi \sin\phi d\theta d\phi d\rho$

4. This problem is about change of variables.
   Consider $\iint_D (2x + y)^2 e^{x-y} dA$ where *D* is region bounded by the lines
   $2x + y = 1$, $2x + y = 4$, $x - y = -1$, and $x - y = 1$.

a. Determine a substitution (u, v) in terms of (x, y) that will that will simplify the double integral.
SOLN:
Let $u = 2x + y$ and
$v = x - y$
so that $x = (u + v)/3$ and $y = (u - 2v)/3$

b. Evaluate the Jacobian for this substitution:
SOLN: $\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{vmatrix} = \frac{-2}{9} - \frac{1}{9} = -\frac{1}{3}$ is negative and so disturbing since it makes an integral which should be positive, negative. So this Jacobian has the wrong sign. This can be fixed by either (i) negating it (!) or (ii) using the cross product vector in the reverse order or (ii) using $v = 2x + y$ and $u = x - y$. Then the Jacobian = 1/3.

c. Convert the integral to one in terms of the variables u and v. Your answer should be a double integral in terms of u and v with specific limits. You don't need to evaluate the integral.

SOLN: $\iint_D (2x+y)^2 e^{x-y} dA = \frac{1}{3}\int_1^4 \int_{-1}^1 u^2 e^v dv du = \frac{1}{3}\int_1^4 u^2 du \int_{-1}^1 e^v dv = \left(\frac{4^3-1}{9}\right)\left(e - \frac{1}{e}\right) \approx 16.4528$

$$\int_{\frac{2}{3}}^{1}\left(\int_{x-1}^{x+1}(2x+y)^2 \exp(x-y)\,dy\right)dx$$

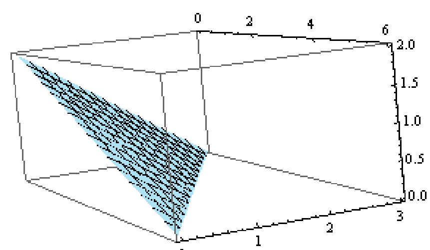
$$+\int_{1}^{\frac{5}{3}}\left(\int_{x-1}^{4-2x}(2x+y)^2 \exp(x-y)\,dy\right)dx$$

$$+\int_{0}^{\frac{2}{3}}\left(\int_{1-2x}^{x+1}(2x+y)^2 \exp(x-y)\,dy\right)dx = 16.4528$$

5. Set up a triple integral find the mass of tetrahedron in the first octant bounded by the plane $x + 2y + 3z = 6$ with mass density distribution is $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

SOLN:
$\int_0^6 \int_0^{3-x/2} \int_0^{2-2y/3-x/3} \sqrt{x^2+y^2+z^2}\,dz\,dy\,d$

This could be easily set up from any of the coordinate planes. In polar coordinates?



6. Consider the ellipsoid $4z^2 + x^2 + y^2 = 4$.

a. Set up an integral for the surface area of the ellipse. Use symmetry, as appropriate.
SOLN: The upper half of the surface can be parameterized as
$\vec{r}(x, y) = \left\langle x, y, \frac{1}{2}\sqrt{4-(x^2+y^2)}\right\rangle$ or $\vec{r}(r,\theta) = \left\langle r, \theta, \frac{1}{2}\sqrt{4-r^2}\right\rangle$

Thus $\left|\vec{r}_x \times \vec{r}_y\right| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \dfrac{-x}{2\sqrt{4-\left(x^2+y^2\right)}} \\ 0 & 1 & \dfrac{-y}{2\sqrt{4-\left(x^2+y^2\right)}} \end{vmatrix} = \sqrt{1+\dfrac{x^2+y^2}{16-4\left(x^2+y^2\right)}} = \dfrac{1}{2}\sqrt{\dfrac{16-3r^2}{4-r^2}}$

Thus $4\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{\dfrac{16-3\left(x^2+y^2\right)}{4-x^2-y^2}}\,dydx = 4\int_0^2 \int_0^{\pi/2} \sqrt{\dfrac{16-3r^2}{4-r^2}}\,rd\theta dr$

b. Use the midpoint rule for double integrals with $m = n = 2$ to approximate this integral.
SOLN: A rectangular domain simplifies the problem. Therefore we prefer the polar form of the integral. Depending on how you do or don't use symmetry, your results may vary here. Partitioning the polar integral above with sample points (½, π/8), (3/2, π/8), (½, 3π/8), (3/2, 3π/8) we have

$$4\int_0^2 \int_0^{\pi/2} \sqrt{\frac{16-3r^2}{4-r^2}}\,rd\theta dr \approx 4(1)\left(\frac{\pi}{2}\right)\left(\frac{1}{2}\sqrt{\frac{16-3/4}{4-1/4}}+\frac{3}{2}\sqrt{\frac{16-27/4}{4-9/4}}\right) = \pi\left(\sqrt{\frac{61}{15}}+3\sqrt{\frac{37}{7}}\right) \approx 28.00$$

Mathematica gives $2\pi\left(4-\frac{\text{Log}[3]}{\sqrt{3}}+\frac{2\text{Log}[3+2\sqrt{3}]}{\sqrt{3}}\right)$ as the value of this integral, which is approximately 34.69